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► To cite this version:

Hacène Djellout, Hui Jiang. LARGE DEVIATIONS OF THE THRESHOLD ESTIMATOR OF INTEGRATED (CO-)VOLATILITY VECTOR IN THE PRESENCE OF JUMPS. *Journal of Theoretical Probability*, 2018, *Journal of Theoretical Probability*, 31 (3), pp.1606-1624. hal-01147189

HAL Id: hal-01147189

<https://hal.science/hal-01147189>

Submitted on 29 Apr 2015

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LARGE DEVIATIONS OF THE THRESHOLD ESTIMATOR OF INTEGRATED (CO-)VOLATILITY VECTOR IN THE PRESENCE OF JUMPS

HACÈNE DJELLOUT AND HUI JIANG

ABSTRACT. Recently a considerable interest has been paid on the estimation problem of the realized volatility and covolatility by using high-frequency data of financial price processes in financial econometrics. Threshold estimation is one of the useful techniques in the inference for jump-type stochastic processes from discrete observations. In this paper, we adopt the threshold estimator introduced by Mancini [18] where only the variations under a given threshold function are taken into account. The purpose of this work is to investigate large and moderate deviations for the threshold estimator of the integrated variance-covariance vector. This paper is an extension of the previous work in Djellout et al [11]. where the problem has been studied in absence of the jump component. We will use the approximation lemma to prove the LDP. As the reader can expect we obtain the same results as in the case without jump.

AMS 2000 subject classifications: 60F10, 62J05, 60J05.

1. MOTIVATION AND CONTEXT

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{[0,1]}, \mathbb{P})$, we consider $X_1 = (X_{1,t})_{t \in [0,1]}$ and $X_2 = (X_{2,t})_{t \in [0,1]}$ two real processes defined by a Lévy jump-diffusion constructed via the superposition of a Wiener process with drift and an independent compound Poisson process. This is one of the first and simplest extensions to the classical geometric Brownian motion underlying the famous Black-Scholes-Merton framework for option pricing.

More precisely, $X_1 = (X_{1,t})_{t \in [0,1]}$ and $X_2 = (X_{2,t})_{t \in [0,1]}$ are given by

$$\begin{cases} dX_{1,t} = b_1(t, \omega)dt + \sigma_{1,t}dW_{1,t} + dJ_{1,t} \\ dX_{2,t} = b_2(t, \omega)dt + \sigma_{2,t}dW_{2,t} + dJ_{2,t} \end{cases} \quad (1.1)$$

for $t \in [0, 1]$ where $W_1 = (W_{1,t})_{t \in [0,1]}$ and $W_2 = (W_{2,t})_{t \in [0,1]}$ are two correlated Wiener processes, with $\rho_t = \text{Cov}(W_{1,t}, W_{2,t}), t \in [0, 1]$. We can write $W_{2,t} = \rho_t dW_{1,t} + \sqrt{1 - \rho_t^2} dW_{3,t}$, where $W_1 = (W_{1,t})_{t \in [0,1]}$ and $W_3 = (W_{3,t})_{t \in [0,1]}$ are independent Wiener processes. J_1 and J_2 are possibly correlated pure jump processes. We assume here that J_1 and J_2 have finite jump activity, that is a.s. there are only finitely many jumps on any finite time interval. A general Lévy model would contain also a compensated infinite activity pure jump component.

Under our assumption J_ℓ is necessarily a compound Poisson process and it can be written as

$$J_{\ell,s} = \sum_{i=1}^{N_{\ell,s}} Y_{\ell,i}, \quad s \in [0, 1].$$

Date: April 29, 2015.

Key words and phrases. Moderate deviation principle, Large deviation principle, Diffusion, Discrete-time observation, Quadratic variation, Realised volatility, Lévy process, Threshold estimator, Jump Poisson.

Here $Y_{\ell,i}$ are i.i.d. real random variables having law ν_ℓ/λ_ℓ , where ν_ℓ is the Lévy measure of X_ℓ normalized by the total mass $\lambda_\ell = \nu_\ell(\mathbb{R} - \{0\}) < +\infty$, and N_ℓ is a poisson process, independent of each $Y_{\ell,i}$, and with constant intensity λ_ℓ .

Such a jump-type stochastic process is recently a standard tool, e.g., for modeling asset values in finance and insurance. The key motivation behind jump-diffusion models is the incorporation of market "stocks", which result in "large" and sudden changes in the price of risky security and which can hardly be modeled by the diffusive component.

In this paper we concentrate on the estimation of

$$[\mathcal{V}]_t = \left(\int_0^t \sigma_{1,s}^2 ds, \int_0^t \sigma_{2,s}^2 ds, \int_0^t \sigma_{1,s} \sigma_{2,s} \rho_s ds \right)$$

Over the last decade, several estimation methods for the integrated variance-covariance \mathcal{V}_t have been proposed. We adopt the threshold estimator which is introduced by Mancini [18] and also by Shimizu and Yoshida [26], independently.

In this method, only the variations under a given threshold function are taken into account. The specific estimator excludes all terms containing jumps from the realized co-variation while remaining consistent, efficient and robust when synchronous data are considered.

Since the seminal work of Mancini [18], several authors have leveraged or extended the thresholding concept to deal with complex stochastic models, see Shimizu and Yoshida [26], or Ogihara and Yoshida [22]. The similar idea is also used by various authors in different contexts; see, e.g., Aït-Sahalia et al. [1], [2] and [3], Gobbi and Mancini [15], Cont and Mancini [21], among others.

So, given the synchronous and evenly-spaced observation of the process $X_{1,t_0}, X_{1,t_1}, \dots, X_{1,t_n}, X_{2,t_0}, X_{2,t_1}, \dots, X_{2,t_n}$ with $t_0 = 0, t_n = 1, n \in \mathbb{N}$, we consider the following statistics

$$\left(\sum_{k=1}^{[nt]} (\Delta_k^n X_1)^2, \sum_{k=1}^{[nt]} (\Delta_k^n X_2)^2, \sum_{k=1}^{[nt]} \Delta_k^n X_1 \Delta_k^n X_2 \right)$$

where $\Delta_k^n X_\ell := X_{\ell,t_k} - X_{\ell,t_{k-1}}$. However this estimate can be highly biased when the processes X_ℓ contain jumps, in fact, as $n \rightarrow \infty$ such a sum approaches the global quadratic variance-covariation

$$([X_1]_t, [X_2]_t, [X_1, X_2]_t)$$

where

$$[X_\ell]_t := \int_0^t \sigma_{\ell,s}^2 ds + \sum_{s \leq t} (\Delta J_{\ell,s})^2, \quad \text{and} \quad [X_1, X_2]_t := \int_0^t \sigma_{1,s} \sigma_{2,s} \rho_s ds + \sum_{s \leq t} \Delta J_{1,s} \Delta J_{2,s}.$$

which also contain the co-jumps, where $\Delta J_{\ell,s} = J_{\ell,s} - J_{\ell,s^-}$.

If we take a deterministic function $r(\frac{1}{n})$ at the step $\frac{1}{n}$ between the observations, such that

$$\lim_{n \rightarrow \infty} r\left(\frac{1}{n}\right) = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log n}{nr\left(\frac{1}{n}\right)} = 0.$$

The function $r(\cdot)$ is a threshold such that whenever $|\Delta_k^n X_\ell|^2 > r(\frac{1}{n})$, a jump has to occur within $[t_{k-1}, t_k]$. Hence we can recover $[\mathcal{V}]_t$ using the following threshold estimator

$$\mathcal{V}_t^n(X) = (\mathcal{Q}_{1,t}^n(X), \mathcal{Q}_{2,t}^n(X), \mathcal{C}_t^n(X))$$

where

$$\mathcal{Q}_{\ell,t}^n(X) = \sum_{k=1}^{[nt]} (\Delta_k^n X_\ell)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell)^2 \leq r(\frac{1}{n})\}}$$

and

$$\mathcal{C}_t^n(X) = \sum_{k=1}^{[nt]} \Delta_k^n X_1 \Delta_k^n X_2 \mathbf{1}_{\{\max_{\ell=1}^2 (\Delta_k^n X_\ell)^2 \leq r(\frac{1}{n})\}}$$

In the work [14], the authors determine what constitutes a good threshold sequence r_n and they propose an objective method for selecting such a sequence.

In the case that X_ℓ have no jumps, this question has been well investigated. The problem of the large deviation of the quadratic estimator of the integrated volatility (without jumps and in the case of synchronous sampling scheme) is obtained in the paper by Djellout et al. [12] and recently Djellout and Samoura [13] have studied the large deviation for the covariance estimator. Djellout et al. [11] have also investigated the problem of the large deviation for the realized (co-)volatility vector which allows them to provide the large deviation for the standard dependence measures between the two assets returns such as the realized regression coefficients, or the realized correlation.

However, the inclusion of jumps within financial models seems to be more and more necessary for practical applications. In this case, Mancini [21] has shown that \mathcal{V}_t^n is a consistent estimators of \mathcal{V}_t and has some asymptotic normality respectively. Furthermore, when $\sigma_t = \sigma$, she [19] studied the large deviation for the threshold estimator. Jiang [16] obtained moderate deviations and functional moderate deviations for threshold estimator. In our paper and by the method as in Mancini [19] and Djellout et al [11], we consider moderate and functional moderate deviation for estimators \mathcal{V}_t^n and large deviation.

More precisely we are interested in the estimations of

$$\mathbb{P} \left(\frac{\sqrt{n}}{v_n} (\mathcal{V}_t^n(X) - [\mathcal{V}]_t) \in A \right)$$

where A is a given domain of deviation, $(v_n)_{n>0}$ is some sequence denoting the scale of deviation. When $v_n = 1$ this is exactly the estimation of central limit theorem. When $v_n = \sqrt{n}$, it becomes the large deviation. Furthermore, when $v_n \rightarrow \infty$ and $v_n = o(\sqrt{n})$, this is the so called moderate deviations. In other words, the moderate deviations investigate the convergence speed between the large deviations and central limit theorem.

Let us recall some basic definitions in large deviations theory. Let $(\mu_t)_{t>0}$ be a family of probability on a topological space (S, \mathcal{S}) where \mathcal{S} is a σ -algebra on S and λ_t be a nonnegative function on $[1, +\infty[$ such that $\lim_{t \rightarrow \infty} \lambda_t = +\infty$. A function $I : S \rightarrow [0, +\infty]$ is said to be a rate function if it is lower semicontinuous and it is said to be a good rate function if its level set $\{x \in S; I(x) \leq a\}$ is a compact for all $a \geq 0$.

(μ_t) is said to satisfy a large deviation principle with speed λ_t and rate function I if for any closed set $F \in \mathcal{S}$

$$\limsup_{t \rightarrow \infty} \frac{1}{\lambda_t} \log \mu_t(F) \leq - \inf_{x \in F} I(x)$$

and for any open set $G \in \mathcal{S}$

$$\limsup_{t \rightarrow \infty} \frac{1}{\lambda_t} \log \mu_t(G) \geq - \inf_{x \in G} I(x).$$

Notations. In the whole paper, for any matrix M , M^T and $\|M\|$ stand for the transpose and the euclidean norm of M , respectively. For any square matrix M , $\det(M)$ is the determinant of M . Moreover, we will shorten large deviation principle by LDP and moderate deviation principle by MDP. We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product. For any process Z_t , $\Delta_s^t Z$ stands for the increment $Z_t - Z_s$. We use $\Delta_k^n Z$ for $\Delta_{t_{k-1}^n}^{t_k^n} Z$. In addition, for a sequence of random variables $(Z_n)_n$ on $\mathbb{R}^{d \times p}$, we say that $(Z_n)_n$ converges (λ_n) -superexponentially fast in probability to some random variable Z if, for all $\delta > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{\lambda_n} \log \mathbb{P}(\|Z_n - Z\| > \delta) = -\infty.$$

This exponential convergence with speed λ_n will be shortened as

$$Z_n \xrightarrow[\lambda_n]{\text{superexp}} Z.$$

The article is arranged in two upcoming sections. Section 2 is devoted to our main results on the LDP and MDP for the (co-)volatility vector in the presence of jumps. In section 3, we give the proof of these theorems.

2. MAIN RESULTS

Let $X_t = (X_{1,t}, X_{2,t})$ be given by (1.1). We introduce the following conditions

(B) for $\ell = 1, 2$ $b(\cdot, \cdot) \in L^\infty(dt \otimes \mathbb{P})$

(LDP) Assume that for $\ell = 1, 2$

- $\sigma_{\ell,t}^2(1 - \rho_t^2)$ and $\sigma_{1,t}\sigma_{2,t}(1 - \rho_t^2) \in L^\infty([0, 1], dt)$.
- the functions $t \rightarrow \sigma_{\ell,t}$ and $t \rightarrow \rho_t$ are continuous.
- let r such that

$$r \left(\frac{1}{n} \right) \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad nr \left(\frac{1}{n} \right) \xrightarrow{n \rightarrow \infty} \infty.$$

(MDP) Assume that for $\ell = 1, 2$

- $\sigma_{\ell,t}^2(1 - \rho_t^2)$ and $\sigma_{1,t}\sigma_{2,t}(1 - \rho_t^2) \in L^2([0, 1], dt)$.
- Let $(v_n)_{n \geq 1}$ be a sequence of positive numbers such that

$$v_n \xrightarrow{n \rightarrow \infty} \infty \quad \text{and} \quad \frac{v_n}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \sqrt{n}v_n r \left(\frac{1}{n} \right) = O(1)$$

$$\text{and for } \ell = 1, 2 \quad \frac{r \left(\frac{1}{n} \right)}{\log \left(\frac{n}{v_n^2} \right) \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} \longrightarrow +\infty. \quad (2.1)$$

We introduce the following function, which will play a crucial role in the calculation of the moment generating function: for $-1 < c < 1$ let for any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$

$$P_c(\lambda) := \begin{cases} -\frac{1}{2} \log \left(\frac{(1 - 2\lambda_1(1 - c^2))(1 - 2\lambda_2(1 - c^2)) - (\lambda_3(1 - c^2) + c)^2}{1 - c^2} \right) & \text{if } \lambda \in \mathcal{D} \\ +\infty, & \text{otherwise} \end{cases} \quad (2.2)$$

where

$$\mathcal{D}_c = \left\{ \lambda \in \mathbb{R}^3, \max_{\ell=1,2} \lambda_\ell < \frac{1}{2(1 - c^2)} \text{ and } \prod_{\ell=1}^2 (1 - 2\lambda_\ell(1 - c^2)) > (\lambda_3(1 - c^2) + c)^2 \right\}. \quad (2.3)$$

Let us present now the main results.

2.1. Moderate deviation. Let us now consider the intermediate scale between the central limit theorem and the law of large numbers.

Theorem 2.1. *For $t=1$ fixed. Under the conditions **(MDP)** and **(B)**, the sequence*

$$\frac{\sqrt{n}}{v_n} (\mathcal{V}_1^n(X) - [\mathcal{V}]_1)$$

satisfies the LDP on \mathbb{R}^3 with speed v_n^2 and with rate function given by

$$I_{mdp}(x) = \sup_{\lambda \in \mathbb{R}^3} \left(\langle \lambda, x \rangle - \frac{1}{2} \langle \lambda, \Sigma_1 \cdot \lambda \rangle \right) = \frac{1}{2} \langle x, \Sigma_1^{-1} \cdot x \rangle \quad (2.4)$$

with

$$\Sigma_1 = \begin{pmatrix} \int_0^1 \sigma_{1,t}^4 dt & \int_0^1 \sigma_{1,t}^2 \sigma_{2,t}^2 \rho_t^2 dt & \int_0^1 \sigma_{1,t}^3 \sigma_{2,t} \rho_t dt \\ \int_0^1 \sigma_{1,t}^2 \sigma_{2,t}^2 \rho_t^2 dt & \int_0^1 \sigma_{2,t}^4 dt & \int_0^1 \sigma_{1,t} \sigma_{2,t}^3 \rho_t dt \\ \int_0^1 \sigma_{1,t}^3 \sigma_{2,t} \rho_t dt & \int_0^1 \sigma_{1,t} \sigma_{2,t}^3 \rho_t dt & \int_0^1 \frac{1}{2} \sigma_{1,t}^2 \sigma_{2,t}^2 (1 + \rho_t^2) dt \end{pmatrix}.$$

Remark 2.1. *Under the condition $b_\ell = 0$, we can prove that for all $\theta \in \mathbb{R}^3$*

$$\lim_{n \rightarrow \infty} \frac{1}{v_n^2} \log \mathbb{E} \left(e^{\sqrt{n} v_n \langle \theta, \mathcal{V}_1^n(X) - [\mathcal{V}]_1 \rangle} \right) = \frac{1}{2} \langle \theta, \Sigma_1 \cdot \theta \rangle.$$

This gives an alternative proof of the moderate deviation using Gärtner-Ellis theorem.

Remark 2.2. *If for some $p > 2$, $\sigma_{1,t}^2$, $\sigma_{2,t}^2$ and $\sigma_{1,t} \sigma_{2,t} (1 - \rho_t^2) \in L^p([0, 1])$ and $v_n = O(n^{\frac{1}{2} - \frac{1}{p}})$, the condition (2.1) in **(MDP)** is verified.*

Let \mathcal{H} be the banach space of \mathbb{R}^3 -valued right-continuous-left-limit non decreasing functions γ on $[0, 1]$ with $\gamma(0) = 0$, equipped with the uniform norm and the σ -field \mathcal{B}^s generated by the coordinate $\{\gamma(t), 0 \leq t \leq 1\}$.

Theorem 2.2. *Under the conditions (MDP) and (B), the sequence*

$$\frac{\sqrt{n}}{v_n} (\mathcal{V}_1^n(X) - [\mathcal{V}].)$$

satisfies the LDP on \mathcal{H} with speed v_n^2 and with rate function given by

$$J_{\text{mdp}}(\phi) = \begin{cases} \int_0^1 \frac{1}{2} \langle \dot{\phi}(t), \bar{\Sigma}_t^{-1} \cdot \dot{\phi}(t) \rangle dt & \text{if } \phi \in \mathcal{AC}_0([0, 1]) \\ +\infty, & \text{otherwise,} \end{cases} \quad (2.5)$$

where

$$\bar{\Sigma}_t = \begin{pmatrix} \sigma_{1,t}^4 & \sigma_{1,t}^2 \sigma_{2,t}^2 \rho_t^2 & \sigma_{1,t}^3 \sigma_{2,t} \rho_t \\ \sigma_{1,t}^2 \sigma_{2,t}^2 \rho_t^2 & \sigma_{2,t}^4 & \sigma_{1,t} \sigma_{2,t}^3 \rho_t \\ \sigma_{1,t}^3 \sigma_{2,t} \rho_t & \sigma_{1,t} \sigma_{2,t}^3 \rho_t & \frac{1}{2} \sigma_{1,t}^2 \sigma_{2,t}^2 (1 + \rho_t^2) \end{pmatrix}$$

is invertible and $\bar{\Sigma}_t^{-1}$ his inverse such that

$$\bar{\Sigma}_t^{-1} = \frac{1}{\det(\bar{\Sigma}_t)} \begin{pmatrix} \frac{1}{2} \sigma_{1,t}^2 \sigma_{2,t}^6 (1 - \rho_t^2) & \frac{1}{2} \sigma_{1,t}^4 \sigma_{2,t}^4 \rho_t^2 (1 - \rho_t^2) & -\sigma_{1,t}^3 \sigma_{2,t}^5 \rho_t (1 - \rho_t^2) \\ \frac{1}{2} \sigma_{1,t}^4 \sigma_{2,t}^4 \rho_t^2 (1 - \rho_t^2) & \frac{1}{2} \sigma_{1,t}^6 \sigma_{2,t}^2 (1 - \rho_t^2) & -\sigma_{1,t}^5 \sigma_{2,t}^3 \rho_t (1 - \rho_t^2) \\ -\sigma_{1,t}^3 \sigma_{2,t}^5 \rho_t (1 - \rho_t^2) & -\sigma_{1,t}^5 \sigma_{2,t}^3 \rho_t (1 - \rho_t^2) & \sigma_{1,t}^4 \sigma_{2,t}^4 (1 - \rho_t^4) \end{pmatrix},$$

$$\text{with } \det(\bar{\Sigma}_t) = \frac{1}{2} \sigma_{1,t}^6 \sigma_{2,t}^6 (1 - \rho_t^2)^3,$$

and $\mathcal{AC}_0 = \{\phi : [0, 1] \rightarrow \mathbb{R}^3 \text{ is absolutely continuous with } \phi(0) = 0\}.$

Remark 2.3. *A similar result for the moderate deviations is obtained by Jiang [16] in the jump case for $\left(\frac{\sqrt{n}}{v_n} \left(\mathcal{Q}_{\ell,t}^n - \int_0^t \sigma_{\ell,s}^2 ds\right)\right)_{n \geq 1}$.*

2.2. Large deviation. Our second result is about the large deviation of $\mathcal{V}_1^n(X)$, i.e. at fixed time.

Theorem 2.3. *Let $t = 1$ be fixed. Under the conditions (LDP) and (B), the sequence $\mathcal{V}_1^n(X)$ satisfies the LDP on \mathbb{R}^3 with speed n and with good rate function given by the legendre transformation of Λ , that is*

$$I_{\text{ldp}}(x) = \sup_{\lambda \in \mathbb{R}^3} (\langle \lambda, x \rangle - \Lambda(\lambda)), \quad (2.6)$$

where $\Lambda(\lambda) = \int_0^1 P_{\rho_t}(\lambda_1 \sigma_{1,t}^2, \lambda_2 \sigma_{2,t}^2, \lambda_3 \sigma_{1,t} \sigma_{2,t}) dt$.

Remark 2.4. Under the condition $b_\ell = 0$, we can calculate the moment generating function of $\mathcal{V}_1^n(X)$. We obtain that for all $\theta = (\theta_1, \theta_2, \theta_3)^T \in \mathcal{D}_{\rho_t}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left(e^{n \langle \theta, \mathcal{V}_1^n(X) \rangle} \right) = \int_0^1 P_{\rho_s} \left(\theta_1 \sigma_{1,s}^2, \theta_2 \sigma_{2,s}^2, \theta_3 \sigma_{1,s} \sigma_{2,s} \right) ds.$$

But the study of the steepness is more difficult.

Let us consider the case where diffusion and correlation coefficients are constant, the rate function being easier to read. Before that let us introduce the function P_c^* which is the Legendre transformation of P_c given in (2.2), for all $x = (x_1, x_2, x_3)$

$$P_c^*(x) := \begin{cases} \log \left(\frac{\sqrt{1-c^2}}{\sqrt{x_1 x_2 - x_3^2}} \right) - 1 + \frac{x_1 + x_2 - 2c x_3}{2(1-c^2)} \\ \quad \text{if } x_1 > 0, x_2 > 0, x_1 x_2 > x_3^2 \\ +\infty, \quad \text{otherwise.} \end{cases} \quad (2.7)$$

Corollary 2.4. We assume that for $\ell = 1, 2$ σ_ℓ and ρ are constants. Under the condition (B), we obtain that $\mathcal{V}_1^n(X)$ satisfies the LDP on \mathbb{R}^3 with speed n and with good rate function $I_{ldp}^\mathcal{V}$ given by

$$I_{ldp}^\mathcal{V}(x_1, x_2, x_3) = P_\rho^* \left(\frac{x_1}{\sigma_1^2}, \frac{x_2}{\sigma_2^2}, \frac{x_3}{\sigma_1 \sigma_2} \right), \quad (2.8)$$

where P_c^* is given in (2.7).

Remark 2.5. In the case σ_ℓ is constant, a similar result for the large deviations is obtained by Mancini [19] in the jump case for $(\mathcal{Q}_{\ell,1}^n)_{n \geq 1}$

Now, we shall extend Theorem 2.3 to the process-level large deviations, i.e. for trajectories $(\mathcal{V}_t^n(X), t \in [0, 1])$ which is interesting from the viewpoint of non-parametric statistics.

Let $\mathcal{BV}([0, 1], \mathbb{R}^3)$ (shorted in \mathcal{BV}) be the space of functions of bounded variation on $[0, 1]$. We identify \mathcal{BV} with $\mathcal{M}_3([0, 1])$, the set of vector measures with value in \mathbb{R}^3 . This is done in the usual manner: to $f \in \mathcal{BV}$, there corresponds μ^f by $\mu^f([0, t]) = f(t)$. Up to this identification, $\mathcal{C}_3([0, 1])$ the set of \mathbb{R}^3 -valued continuous bounded functions on $[0, 1]$, is the topology dual of \mathcal{BV} . We endow \mathcal{BV} with the weak-* convergence topology $\sigma(\mathcal{BV}, \mathcal{C}_3([0, 1]))$ and with the associated Borel- σ -field \mathcal{B}_ω . Let $f \in \mathcal{BV}$ and μ^f the associated measure in $\mathcal{M}_3([0, 1])$. Consider the Lebesgue decomposition of μ^f , $\mu^f = \mu_a^f + \mu_s^f$ where μ_a^f denotes the absolutely continuous part of μ^f with respect to dx and μ_s^f its singular part. We denote by $f_a(t) = \mu_a^f([0, t])$ and by $f_s(t) = \mu_s^f([0, t])$.

Theorem 2.5. Under the conditions (LDP) and (B), the sequence $\mathcal{V}_1^n(X)$ satisfies the LDP on \mathcal{BV} with speed n and rate function J_{ldp} given for any $f = (f_1, f_2, f_3) \in \mathcal{BV}$ by

$$J_{ldp}(f) = \int_0^1 P_{\rho_t}^* \left(\frac{f'_{1,a}(t)}{\sigma_{1,t}^2}, \frac{f'_{2,a}(t)}{\sigma_{2,t}^2}, \frac{f'_{3,a}(t)}{\sigma_{1,t} \sigma_{2,t}} \right) + \int_0^1 \frac{\sigma_{2,t}^2 f'_{1,s}(t) + \sigma_{1,t}^2 f'_{2,s}(t) - 2\rho_t \sigma_{1,t} \sigma_{2,t} f'_{3,s}(t)}{2\sigma_{1,t}^2 \sigma_{2,t}^2 (1 - \rho_t^2)} \mathbf{1}_{[t: f'_{1,s} > 0, f'_{2,s} > 0, (f'_{3,s})^2 < f'_{1,s} f'_{2,s}]} d\theta(t), \quad (2.9)$$

where P_c^* is given in (2.7) and θ is any real-valued nonnegative measure with respect to which μ_s^f is absolutely continuous and $f'_s = d\mu_s^f/d\theta = (f'_{1,s}, f'_{2,s}, f'_{3,s})$.

3. PROOFS

For the convenience of the reader, we recall the following lemma which is the key of the proofs.

Lemma 3.1. (*Approximation Lemma*) *Theorem 4.2.13 in [10]*

Let $(Y^n, X^n, n \in \mathbb{N})$ be a family of random variables valued in a Polish space S with metric $d(\cdot, \cdot)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume

- $\mathbb{P}(Y^n \in \cdot)$ satisfies the large deviation principle with speed ϵ_n ($\epsilon_n \rightarrow \infty$) and the good rate function I .
- for every $\delta > 0$

$$\limsup_{n \rightarrow \infty} \frac{1}{\epsilon_n} \log \mathbb{P}(d(Y^n, X^n) > \delta) = -\infty.$$

Then $\mathbb{P}(X^n \in \cdot)$ satisfies the large deviation principle with speed ϵ_n and the good rate function I .

Before starting the proof, we need to introduce some technical tools. In the case without jumps, we introduce the following diffusion for $\ell = 1, 2$

$$D_{\ell,t} = \int_0^t \sigma_{\ell,s} dW_{\ell,s},$$

where $W_{\ell,s}$ and $\sigma_{\ell,s}$ are defined as before. We introduce the correspondent estimator

$$V_t^n = (Q_{1,t}^n, Q_{2,t}^n, C_t^n)$$

where for $\ell = 1, 2$

$$Q_{\ell,t}^n = \sum_{k=1}^{[nt]} (\Delta_k^n D_\ell)^2 \quad \text{and} \quad C_t^n = \sum_{k=1}^{[nt]} \Delta_k^n D_1 \Delta_k^n D_2.$$

We recall the following results from Djellout et al. [11]

Proposition 3.2. *Under the conditions (B) and (MDP),*

(1) *the sequence*

$$\frac{\sqrt{n}}{v_n} (V_1^n - [\mathcal{V}]_1)$$

satisfies the LDP on \mathbb{R}^3 with speed v_n^2 and with rate function given by (2.1).

(2) *the sequence*

$$\frac{\sqrt{n}}{v_n} (V^n - [\mathcal{V}].)$$

satisfies the LDP on \mathcal{H} with speed v_n^2 and with rate function given by (2.2).

Proposition 3.3. *Under the conditions (B) and (LDP),*

(1) *the sequence V_1^n satisfies the LDP on \mathbb{R}^3 with speed n and with good rate function given in (2.6).*

(2) *the sequence \mathcal{V}^n satisfies the LDP on \mathcal{BV} with speed n and rate function J_{ldp} given by (2.9).*

3.1. Proof of Theorem 2.1.

We will do the proof in two steps.

Part 1 We start with the case $b_\ell = 0$. In this case, $\mathcal{V}_t^n(X) = \mathcal{V}_t^n(X^0)$ with $X_{\ell,t}^0 = X_{\ell,t} - \int_0^t b_\ell(s, \omega) ds$ and

$$\mathcal{Q}_{\ell,1}^n(X^0) = \sum_{k=1}^n (\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}}, \quad \ell = 1, 2$$

and

$$\mathcal{C}_1^n(X^0) = \sum_{k=1}^n \Delta_k^n X_1^0 \Delta_k^n X_2^0 \mathbf{1}_{\{\max_{\ell=1}^2 (\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}}.$$

We will prove that

$$\frac{\sqrt{n}}{v_n} (\mathcal{V}_1^n(X^0) - V_1^n) \xrightarrow[v_n^2]{\text{superexp}} 0.$$

For that, we will prove that for $\ell = 1, 2$

$$\frac{\sqrt{n}}{v_n} (\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n) \xrightarrow[v_n^2]{\text{superexp}} 0, \quad (3.1)$$

and

$$\frac{\sqrt{n}}{v_n} (\mathcal{C}_1^n(X^0) - C_1^n) \xrightarrow[v_n^2]{\text{superexp}} 0. \quad (3.2)$$

We start by the proof of (3.1). Since the processes X_ℓ^0 and D_ℓ have independent increment, by Chebyshev inequality we obtain for all $\theta > 0$

$$\mathbb{P} \left(\frac{\sqrt{n}}{v_n} (\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n) > \delta \right) \leq e^{-\theta \delta v_n^2} \prod_{k=1}^n \mathbb{E} \left(e^{\theta \sqrt{n} v_n [(\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} - (\Delta_k^n D_\ell)^2]} \right).$$

We have to control each term appearing in the product

$$\mathbb{E} \left(e^{\theta \sqrt{n} v_n [(\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} - (\Delta_k^n D_\ell)^2]} \right) \leq \mathfrak{R}_1(k, n) + \mathfrak{R}_2(k, n), \quad (3.3)$$

where

$$\mathfrak{R}_1(k, n) := \mathbb{E} \left(e^{\theta \sqrt{n} v_n [(\Delta_k^n X_\ell^0)^2 - (\Delta_k^n D_\ell)^2]} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} \right)$$

and

$$\mathfrak{R}_2(k, n) := \mathbb{P} \left((\Delta_k^n X_\ell^0)^2 > r(\frac{1}{n}) \right).$$

For the first term, we write

$$\begin{aligned} \mathfrak{R}_1(k, n) &= \mathbb{E} \left(e^{\theta \sqrt{n} v_n [(\Delta_k^n X_\ell^0)^2 - (\Delta_k^n D_\ell)^2]} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} \mid \Delta_k^n N_\ell = 0 \right) \mathbb{P}(\Delta_k^n N_\ell = 0) \\ &\quad + \mathbb{E} \left(e^{\theta \sqrt{n} v_n [(\Delta_k^n X_\ell^0)^2 - (\Delta_k^n D_\ell)^2]} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n}), \Delta_k^n N_\ell \neq 0\}} \right). \end{aligned} \quad (3.4)$$

Since N_ℓ is independent of W_ℓ , we obtain that

$$\begin{aligned} \mathfrak{R}_1(k, n) &\leq \mathbb{P} \left((\Delta_k^n D_\ell)^2 \leq r(\frac{1}{n}) \right) e^{-\lambda_\ell/n} + e^{\sqrt{n} v_n \theta r(\frac{1}{n})} (1 - e^{-\lambda_\ell/n}) \\ &\leq 1 + e^{\sqrt{n} v_n \theta r(\frac{1}{n})} (1 - e^{-\lambda_\ell/n}). \end{aligned} \quad (3.5)$$

Now we have to control $\mathfrak{R}_2(k, n)$, by the same argument as before we have

$$\begin{aligned}\mathfrak{R}_2(k, n) &= \mathbb{P} \left((\Delta_k^n X_\ell^0)^2 > r \left(\frac{1}{n} \right) \mid \Delta_k^n N_\ell = 0 \right) \mathbb{P}(\Delta_k^n N_\ell = 0) \\ &\quad + \mathbb{P} \left((\Delta_k^n X_\ell^0)^2 > r \left(\frac{1}{n} \right), \Delta_k^n N_\ell \neq 0 \right) \\ &\leq \mathbb{P} \left((\Delta_k^n D_\ell)^2 > r \left(\frac{1}{n} \right) \right) e^{-\lambda_\ell/n} + (1 - e^{-\lambda_\ell/n}).\end{aligned}$$

From exponential inequality for martingales, it follows that for $\ell = 1, 2$,

$$\mathbb{P} \left((\Delta_k^n D_\ell)^2 > r \left(\frac{1}{n} \right) \right) \leq \exp \left(- \frac{r \left(\frac{1}{n} \right)}{2 \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} \right), \quad (3.6)$$

which implies that

$$\mathfrak{R}_2(k, n) \leq \exp \left(- \frac{r \left(\frac{1}{n} \right)}{2 \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} \right) + (1 - e^{-\lambda_\ell/n}). \quad (3.7)$$

From (3.3), (3.5) and (3.7), we obtain that

$$\begin{aligned}\mathbb{E} \left(e^{\theta \sqrt{n} v_n \left[(\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r \left(\frac{1}{n} \right)\}} - (\Delta_k^n D_\ell)^2 \right]} \right) &\leq 1 + (1 + e^{\sqrt{n} v_n \theta r \left(\frac{1}{n} \right)}) (1 - e^{-\lambda_\ell/n}) \\ &\quad + \exp \left(- \frac{r \left(\frac{1}{n} \right)}{2 \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} \right).\end{aligned}$$

Using the hypotheses **(MDP)**, we have

$$\limsup_{n \rightarrow \infty} \frac{n}{v_n^2} \max_{k=1}^n \log \mathbb{E} \left(e^{\theta \sqrt{n} v_n \left[(\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r \left(\frac{1}{n} \right)\}} - (\Delta_k^n D_\ell)^2 \right]} \right) = 0. \quad (3.8)$$

So

$$\limsup_{n \rightarrow \infty} \frac{1}{v_n^2} \log \mathbb{P} \left(\frac{\sqrt{n}}{v_n} (\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n) > \delta \right) \leq -\lambda \delta.$$

Letting λ goes to infinity, we obtain that the right hand of the last inequality goes to $-\infty$. Proceeding in the same way for $-(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n)$ we obtain (3.1).

Now we have to prove (3.2). For that we have the following decomposition

$$C_1^n(X^0) - C_1^n = \frac{1}{2} [\overline{\mathcal{Q}}_{3,1}^n(X^0) - Q_{3,1}^n] - \frac{1}{2} \left[\sum_{\ell=1}^2 \overline{\mathcal{Q}}_{\ell,1}^n(X^0) - Q_{\ell,1}^n \right], \quad (3.9)$$

where

$$Q_{3,1}^n = \sum_{k=1}^n (\Delta_k^n D_1 + \Delta_k^n D_2)^2,$$

and for $\ell = 1, 2$

$$\overline{\mathcal{Q}}_{\ell,t}^n(X^0) = \sum_{k=1}^n (\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{\max_{\ell=1}^2 (\Delta_k^n X_\ell^0)^2 \leq r \left(\frac{1}{n} \right)\}}$$

and

$$\overline{\mathcal{Q}}_{3,1}^n(X^0) = \sum_{k=1}^n (\Delta_k^n X_1^0 + \Delta_k^n X_2^0)^2 \mathbf{1}_{\{\max_{\ell=1}^2 (\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}}.$$

Remark that $\overline{\mathcal{Q}}_{\ell,t}^n(X^0)$ is a slight modification of $\mathcal{Q}_{\ell,t}^n(X^0)$.

We know that $\Delta_k^n D_1 + \Delta_k^n D_2 \sim \mathcal{N}(0, \beta^2(k, n))$ with

$$\beta^2(k, n) = \int_{t_{k-1}}^{t_k} \sigma_{1,s}^2 ds + \int_{t_{k-1}}^{t_k} \sigma_{2,s}^2 ds + 2 \int_{t_{k-1}}^{t_k} \sigma_{1,s} \sigma_{2,s} \rho_s ds.$$

For all $\delta > 0$, we have

$$\mathbb{P} \left(\frac{\sqrt{n}}{v_n} |\mathcal{C}_1^n(X^0) - C_1^n| > \delta \right) \leq 3 \max_{\ell=1}^3 \mathbb{P} \left(\frac{\sqrt{n}}{v_n} |\overline{\mathcal{Q}}_{\ell,1}^n(X^0) - Q_{\ell,1}^n| > \frac{2\delta}{3} \right).$$

So we obtain (3.2).

Part 2 We have to prove that

$$\frac{\sqrt{n}}{v_n} (\mathcal{V}_1^n(X) - \mathcal{V}_1^n(X^0)) \xrightarrow[v_n^2]{\text{superexp}} 0.$$

We have that

$$|\mathcal{Q}_{\ell,1}^n(X) - \mathcal{Q}_{\ell,1}^n(X^0)| \leq \varepsilon(n) \mathcal{Q}_{\ell,1}^n(X^0) + \left(1 + \frac{1}{\varepsilon(n)}\right) Z_\ell^n \quad (3.10)$$

and

$$|\mathcal{C}_1^n(X) - \mathcal{C}_1^n(X^0)| \leq \varepsilon(n) \max_{\ell=1}^2 \mathcal{Q}_{\ell,1}^n(X^0) + \left(1 + \frac{1}{\varepsilon(n)}\right) \max_{\ell=1}^2 Z_\ell^n, \quad (3.11)$$

where

$$Z_\ell^n = \sum_{k=1}^n \left(\int_{t_{k-1}}^{t_k} b_\ell(s, \omega) ds \right)^2.$$

By the condition **(B)**, we have that $\|Z_\ell^n\| \leq \frac{1}{n}$. We choose $\varepsilon(n)$ such that

$$\frac{\sqrt{n}}{v_n} \varepsilon(n) \rightarrow 0, \quad v_n \sqrt{n} \varepsilon(n) \rightarrow \infty,$$

so by the MDP of $\mathcal{Q}_{\ell,1}^n(X^0)$, we obtain the result.

3.2. Proof of Theorem 2.2.

Since the sequence $\frac{\sqrt{n}}{v_n}(V^n - [\mathcal{V}])$ satisfies the LDP on \mathcal{H} with speed v_n^2 and rate function J_{mdp} , by Lemma 3.1, it is sufficient to show that:

$$\frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \|\mathcal{V}_t^n(X^0) - V_t^n\| \xrightarrow[v_n^2]{\text{superexp}} 0. \quad (3.12)$$

Lemma 3.4. *Under the condition **(MDP)**, we have*

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \|\mathbb{E} \mathcal{V}_t^n(X^0) - [\mathcal{V}]_t\| = 0.$$

Proof We will prove that for $\ell = 1, 2$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left| \mathbb{E} \mathcal{Q}_{\ell,t}^n(X^0) - \int_0^t \sigma_{\ell,s}^2 ds \right| = 0. \quad (3.13)$$

and

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left| \mathbb{E} \mathcal{C}_t^n(X^0) - \int_0^t \sigma_{1,s} \sigma_{1,s} \rho_s ds \right| = 0. \quad (3.14)$$

In fact, (3.13) can be done in the same way as in Jiang [16]. It remains to show (3.14). Using (3.9), we obtain that

$$\left| \mathbb{E} \mathcal{C}_t^n(X^0) - \int_0^t \sigma_{1,s} \sigma_{1,s} \rho_s ds \right| \leq \frac{1}{2} \left| \mathbb{E} \overline{\mathcal{Q}}_{3,t}^n(X^0) - \beta_t \right| + \max_{\ell=1}^2 \left| \mathbb{E} \overline{\mathcal{Q}}_{\ell,t}^n(X^0) - \int_0^t \sigma_{\ell,s}^2 ds \right|,$$

where $\beta_t = \int_0^t \sigma_{1,s}^2 ds + \int_0^t \sigma_{2,s}^2 ds + 2 \int_0^t \sigma_{1,s} \sigma_{2,s} \rho_s ds$. So the proof of (3.14) is a consequence of (3.13) and the fact that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left| \mathbb{E} \overline{\mathcal{Q}}_{3,t}^n(X^0) - \beta_t \right| = 0,$$

which is an adaptation of the proof in Jiang [16].

Proof of Theorem 2.2

For (3.12), we will prove that for $\ell = 1, 2$

$$\frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left\| \mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n \right\| \xrightarrow[v_n^2]{\text{superexp}} 0 \quad \text{and} \quad \frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left\| \mathcal{C}_t^n(X^0) - C_t^n \right\| \xrightarrow[v_n^2]{\text{superexp}} 0.$$

From Lemma 3.4, it follows that as $n \rightarrow \infty$

$$\frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left(\mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n) \vee \mathbb{E}(\mathcal{C}_t^n(X^0) - C_t^n) \right) \rightarrow 0. \quad (3.15)$$

Then, we only need to prove that

$$\frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left\| \mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n - \mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n) \right\| \xrightarrow[v_n^2]{\text{superexp}} 0 \quad (3.16)$$

and

$$\frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left\| \mathcal{C}_t^n(X^0) - C_t^n - \mathbb{E}(\mathcal{C}_t^n(X^0) - C_t^n) \right\| \xrightarrow[v_n^2]{\text{superexp}} 0. \quad (3.17)$$

We start by the proof of (3.16). Remark that $(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n - \mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n))$ is a $\mathcal{F}_{[nt]/n}$ -martingale. Then

$$\exp \left(\lambda \left(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n - \mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n) \right) \right)$$

is a submartingale. By the maximal inequality, we have for any $\eta, \lambda > 0$

$$\begin{aligned} & \mathbb{P} \left(\frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} \left(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n - \mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n) \right) > \eta \right) \\ & \leq e^{-\lambda v_n^2 \eta} \mathbb{E} \exp \left(\lambda \sqrt{n} v_n \left(\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n - \mathbb{E}(\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n) \right) \right) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P} \left(\frac{\sqrt{n}}{v_n} \inf_{t \in [0,1]} (\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n - \mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n)) < -\eta \right) \\ & \leq e^{-\lambda v_n^2 \eta} \mathbb{E} \exp \left(-\lambda \sqrt{n} v_n (\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n - \mathbb{E}(\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n)) \right). \end{aligned}$$

Together with (3.8) and (3.15), we have

$$\limsup_{n \rightarrow \infty} \frac{1}{v_n^2} \log \mathbb{P} \left(\frac{\sqrt{n}}{v_n} \sup_{t \in [0,1]} |\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n - \mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n)| > \eta \right) \leq -\lambda \eta.$$

(3.16) can be obtained by letting λ goes to infinity.

Similarly, we can have (3.17) by (3.8), (3.9) and (3.15).

3.3. Proof of Theorem 2.3.

We will do the proof in two steps.

Step 1 We will prove that

$$\mathcal{V}_1^n(X^0) - V_1^n \xrightarrow[n]{\text{superexp}} 0.$$

For that, we will prove that for $\ell = 1, 2$

$$\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n \xrightarrow[n]{\text{superexp}} 0, \quad (3.18)$$

and

$$\mathcal{C}_1^n(X^0) - C_1^n \xrightarrow[n]{\text{superexp}} 0. \quad (3.19)$$

We start by the proof of (3.18). Since the processes X_ℓ and D_ℓ have independent increment, by Chebyshev inequality we obtain for all $\theta > 0$

$$\mathbb{P}(\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n > \delta) \leq e^{-\theta n \delta} \prod_{k=1}^n \mathbb{E} \left(e^{\theta n \left[(\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} - (\Delta_k^n D_\ell)^2 \right]} \right).$$

Similar to (3.3),

$$\mathbb{E} \left(e^{\theta n \left[(\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} - (\Delta_k^n D_\ell)^2 \right]} \right) \leq I_1(k, n) + I_2(k, n),$$

where

$$I_1(k, n) := \mathbb{E} \left(e^{\theta n \left[(\Delta_k^n X_\ell^0)^2 - (\Delta_k^n D_\ell)^2 \right]} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} \right)$$

and

$$I_2(k, n) := \mathbb{P} \left((\Delta_k^n X_\ell^0)^2 > r(\frac{1}{n}) \right)$$

From (3.4), (3.5) and (3.7), it follows that

$$I_2(k, n) \leq \exp \left(-\frac{r(\frac{1}{n})}{2 \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} \right) + (1 - e^{-\lambda_\ell/n}).$$

and

$$I_1(k, n) \leq 1 + \mathbb{E} \left(e^{\theta n \left[(\Delta_k^n X_\ell^0)^2 - (\Delta_k^n D_\ell)^2 \right]} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n}), \Delta_k^n N_\ell \neq 0\}} \right).$$

Let (α_n) be a sequence of real numbers such that $\alpha_n \rightarrow 0$, which will be chosen latter. We have

$$\mathbb{E} \left(e^{\theta n (\Delta_k^n X_\ell^0)^2} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n}), \Delta_k^n N_\ell \neq 0\}} \right) = F_1(k, n) + F_2(k, n),$$

where

$$F_1(k, n) := \mathbb{E} \left(e^{\theta n (\Delta_k^n X_\ell^0)^2} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n}), \Delta_k^n N_\ell \neq 0, |\Delta_k^n J_\ell| \leq \alpha_n\}} \right)$$

and

$$F_2(k, n) := \mathbb{E} \left(e^{\theta n (\Delta_k^n X_\ell^0)^2} \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n}), \Delta_k^n N_\ell \neq 0, |\Delta_k^n J_\ell| > \alpha_n\}} \right).$$

We have to prove that for $\ell = 1, 2$ $\lim_{n \rightarrow \infty} \max_{k=1}^n F_\ell(k, n) \rightarrow 0$. We start with $F_2(k, n)$. From condition **(LDP)**, it follows that $n \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds < +\infty$.

So for all $\theta > 0$, we choose

$$\alpha_n = \left(2 \sqrt{\theta n \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} + 1 \right) \sqrt{r(1/n)}.$$

Then it is easy to see that

$$F_2(k, n) \leq e^{\theta n r(\frac{1}{n})} \mathbb{P} \left(|Z| \geq \frac{2 \sqrt{\theta n \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} \sqrt{r(\frac{1}{n})}}{\sqrt{\int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds}} \right),$$

where Z is a standard Gaussian random variable. As a consequence of the well-known inequality $\int_y^{+\infty} e^{-\frac{z^2}{2}} dz \leq (1/y) e^{-\frac{y^2}{2}}$, for all $y > 0$, we obtain

$$F_2(k, n) \leq e^{\theta n r(\frac{1}{n})} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\theta n r(1/n)}} e^{-2\theta n r(\frac{1}{n})}.$$

So for n large enough and $\theta > 1$, we have

$$\max_{k=1}^n F_2(k, n) \leq e^{-\theta n r(\frac{1}{n})} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now we will control $F_1(k, n)$. Using the fact that

$$\theta n (\Delta_k^n X_\ell^0)^2 \leq \theta n \left[\frac{1}{4\theta n \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds} (\Delta_k^n D_\ell)^2 + 4\theta n \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds (\Delta_k^n J_\ell)^2 \right],$$

we have with the same choose of the sequence α_n , by independence of $\Delta_k^n D_\ell$ and $\Delta_k^n J_\ell$ and Cauchy-Schwarz inequality that

$$\begin{aligned} F_1(k, n) &\leq \mathbb{E} \left(e^{\frac{(\Delta_k^n D_\ell)^2}{4 \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds}} \right) \mathbb{E} \left(e^{4\theta^2 \left(n \max_{k=1}^n \int_{t_{k-1}}^{t_k} \sigma_{\ell,s}^2 ds \right) n (\Delta_k^n J_\ell)^2} \mathbf{1}_{\{|\Delta_k^n J_\ell| \leq \alpha_n\}} \mathbf{1}_{\{\Delta_k^n N_\ell \neq 0\}} \right) \\ &\leq \mathbb{E} \left(e^{\frac{Z^2}{4}} \right) \mathbb{E}^{\frac{1}{2}} \left(e^{8\theta^2 n (\Delta_k^n J_\ell)^2} \mathbf{1}_{\{|\Delta_k^n J_\ell| \leq \alpha_n\}} \right) \mathbb{P}^{\frac{1}{2}} (\Delta_k^n N_\ell \neq 0). \end{aligned}$$

From Mancini [19] page 877, we conclude that

$$\lim_{n \rightarrow \infty} \max_{k=1}^n \mathbb{E} \left(e^{8\theta^2 n (\Delta_k^n J_\ell)^2} \mathbf{1}_{\{|\Delta_k^n J_\ell| \leq \alpha_n\}} \right) < \infty.$$

Since Z is a standard Gaussian random variable, we conclude that

$$E \left(e^{\frac{Z^2}{4}} \right) < \infty.$$

So that $\max_{k=1}^n F_1(k, n) \leq C(1 - e^{-\lambda_\ell/n}) \rightarrow 0$ as $n \rightarrow \infty$.

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{k=1}^n \mathbb{E} \left(e^{\theta n \left[(\Delta_k^n X_\ell^0)^2 \mathbf{1}_{\{(\Delta_k^n X_\ell^0)^2 \leq r(\frac{1}{n})\}} - (\Delta_k^n D_\ell)^2 \right]} \right) = 0,$$

which implies that for any $\theta > 1$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P} \left(\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n > \delta \right) \leq -\theta\delta.$$

Letting θ goes to infinity, we obtain that the left term in the last inequality goes to $-\infty$. And similarly, by doing the same calculation with

$$\mathbb{P} \left(\mathcal{Q}_{\ell,1}^n(X^0) - Q_{\ell,1}^n < -\delta \right),$$

we can get (3.18).

To prove (3.19), we use the decomposition (3.9) and an adaptation of the proof of (3.18).

Step 2 We will prove that

$$\mathcal{V}_1^n(X) - \mathcal{V}_1^n(X^0) \xrightarrow[n]{\text{superexp}} 0.$$

For that we use (3.10) and (3.11) and we choose $\varepsilon(n)$ such that $n\varepsilon(n) \rightarrow 0$ to obtain the result.

3.4. Proof of Theorem 2.5.

We will prove that for $\ell = 1, 2$

$$\sup_{t \in [0,1]} \left\| \mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n \right\| \xrightarrow[n]{\text{superexp}} 0 \quad \text{and} \quad \sup_{t \in [0,1]} \left\| \mathcal{C}_t^n(X^0) - C_t^n \right\| \xrightarrow[n]{\text{superexp}} 0.$$

To do that we use the same argument as in the proof of Theorem 2.2 and the fact that

$$\sup_{t \in [0,1]} \left| \mathbb{E}(\mathcal{Q}_{\ell,t}^n(X^0) - Q_{\ell,t}^n) \right| \rightarrow 0.$$

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